

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
SOLUTION SKETCHES TO HOMEWORK 7

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Homework 7.1 (A general version of Bloch's theorem). Show if $f: U \rightarrow \mathbb{C}$ is holomorphic and $f'(c) \neq 0$ at a point $c \in U$, then $f(U)$ contains disks of every radius $(3/2 - \sqrt{2})s|f'(c)|$, where $s \in (0, \text{dist}(c, \partial U))$. In particular, show if $f: \mathbb{C} \rightarrow \mathbb{C}$ is entire and nonconstant, then $f(\mathbb{C})$ contains disks of arbitrarily large radii.

Solution. The function $g: B_1(0) \rightarrow \mathbb{C}$ defined by

$$g(z) = \frac{f(sz + c)}{sf'(c)}$$

is well-defined and can be extended holomorphically to the slightly larger ball $B_{1+\eta}(0)$ for $0 < \eta < (\text{dist}(c, \partial U) - s)/s$. Therefore, $g \in \mathcal{H}(\bar{B}_1(0))$ and by construction $g'(0) = 1$. Applying Bloch's theorem from the lecture notes yields the existence of $p \in \mathbb{C}$ such that $B_{3/2-\sqrt{2}}(p) \subset g(B_1(0))$. It follows that

$$sf'(c)B_{3/2-\sqrt{2}}(p) \subset f(B_s(c)) \subset f(U).$$

This proves the first statement.

When f is entire and nonconstant, there exists $c \in \mathbb{C}$ such that $f'(c) \neq 0$. Thus we can apply the first statement for every $s > 0$ which proves the second assertion.

Homework 7.2 (Biholomorphic functions on the unit ball). Let $z_0 \in B_1(0)$ and define the function $\varphi_{z_0}: B_1(0) \rightarrow \mathbb{C}$ by

$$\varphi_{z_0}(z) := \frac{z - z_0}{1 - \bar{z}_0 z}.$$

- Show φ_{z_0} is a biholomorphic¹ map from $B_1(0)$ to $B_1(0)$.
- Show every biholomorphic map $f: B_1(0) \rightarrow B_1(0)$ is of the form $f = a \varphi_{z_0}$, where $a \in \partial B_1(0)$ ².

Solution. a. Since $|\bar{z}_0 z| = |z_0||z| < |z|$, the denominator never vanishes for $z \in B_1(0)$, showing φ_{z_0} is holomorphic.

We next check that φ_{z_0} indeed maps into the unit disk $B_1(0)$. Indeed, the inequality $|\varphi_{z_0}(z)| < 1$ holds if and only if

$$|z|^2 - z\bar{z}_0 - \bar{z}z_0 + |z_0|^2 < 1 - \bar{z}_0 z - z_0 \bar{z} + |\bar{z}_0 z|^2.$$

Canceling terms we see the claim $|\varphi_{z_0}(z)| < 1$ is equivalent to

$$|z|^2 + |z_0|^2 < 1 + |z|^2 |z_0|^2.$$

Since $|z_0| < 1$ this is always true for $z \in B_1(0)$.³ To show that φ_{z_0} is bijective it suffices to note that the inverse function is given by φ_{-z_0} .

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¹Recall a map is biholomorphic if it is bijective, holomorphic, and its inverse is holomorphic.

²**Hint.** Recall the Schwarz lemma.

³Consider the real-squared-to-real assignment $f(x, y) := x + y - xy$. Clearly, it has a maximum on $[0, 1]^2$. As $\nabla f(x, y) = (1 - y)e_1 + (1 - x)e_2$ its gradient never vanishes in the interior $(0, 1)^2$, so that the maximum lies on the boundary. Hence either $x \in \{0, 1\}$ or $y \in \{0, 1\}$. In both cases one can check $f(x, y) \leq 1$. Thus $f(x, y) < 1$ for $(x, y) \in (0, 1)^2$.

b. In order to prove the representation of biholomorphic maps from $B_1(0)$ to $B_1(0)$ we note that, setting $-z_0 = f^{-1}(0)$, the function $f \circ \varphi_{z_0}$ satisfies $f(\varphi_{z_0}(0)) = 0$. Hence by the Schwarz lemma we get $|f(\varphi_{z_0}(z))| \leq |z|$. Similarly, for the inverse $\varphi_{z_0}^{-1} \circ f^{-1}$ we deduce $|\varphi_{z_0}^{-1}(f^{-1}(y))| \leq |y|$. Inserting $y = f(\varphi_{z_0}(z))$ we obtain that $|f(\varphi_{z_0}(z))| = |z|$, so that again by the Schwarz Lemma there exists a constant $a \in \partial B_1(0)$ such that $f(\varphi_{z_0}(z)) = az$ for every $z \in B_1(0)$. Hence $f(z) = a\varphi_{-z_0}(z)$ for every $z \in B_1(0)$.

Homework 7.3 (Improvement of the Bloch constant). The purpose of this exercise is to improve the constant $3/2 - \sqrt{2}$ appearing in Bloch's theorem⁴. We will show that for every $f \in \mathcal{H}(\bar{B}_1(0))$ with $f'(0) = 1$ the image $f(B_1(0))$ contains a disk of radius $3\sqrt{2}/2 - 2$.

- a. Motivated by Homework 7.1, we look for a function $F \in \mathcal{H}(\bar{B}_1(0))$ such that $f(B_1(0)) = F(B_1(0))$ with maximal value $|F'(0)|$. To make this precise, set

$$\mathcal{F} := \{f \circ (a\varphi_{z_0}) : z_0 \in B_1(0), a \in \partial B_1(0)\},$$

where φ_{z_0} is from Homework 7.2. Show that every $h \in \mathcal{F}$ with $h = f \circ (a\varphi_{z_0})$ obeys $h \in \mathcal{H}^1(\bar{B}_1(0))$, $h(B_1(0)) = f(B_1(0))$ and $|h'(0)| = |f'(-az_0)|(1 - |z_0|^2)$.

- b. Let q denote the maximizer of the assignment $|f'(z)|(1 - |z|^2)$, where $z \in \bar{B}_1(0)$. Show $q \in B_1(0)$ and that, setting $F := f \circ (\varphi_{-q})$, we have

$$|F'(z)| \leq \frac{|f'(q)|(1 - |q|^2)}{1 - |z|^2}.$$

- c. Deduce $|F'(z)| \leq 2|F'(0)|$ whenever $|z| \leq \sqrt{2}/2$. Conclude the proof using Step 2 of the proof of Bloch's theorem in the lecture.

Solution. a. Each function φ_{z_0} in question has its only singularity in $\hat{z} = z_0/|z_0|^2$. Note that $|\hat{z}| = |z_0|^{-1} > 1$, which yields $\varphi_{z_0} \in \mathcal{H}(\bar{B}_1(0))$. In order to show the composition $h := f \circ (a\varphi_{z_0})$ belongs to $\mathcal{H}(\bar{B}_1(0))$, we have to ensure for every $r > 0$ there exists $\eta > 0$ such that $a\varphi_{z_0}(B_{1+\eta}(0)) \subset B_{1+r}(0)$, which implies holomorphy of the composition on $B_{1+\eta}(0)$. Since $|a| = 1$ it suffices to consider $a = 1$. We argue by contradiction and assume that there exists $r > 0$ such that for every $n \in \mathbb{N}$, setting $\eta_n = 1/n$ there is $z_n \in B_{1+\eta_n}(0)$ with $|\varphi_{z_0}(z_n)| \geq 1 + r$. Up to a subsequence we know $(z_n)_{n \in \mathbb{N}}$ converges to some $z_\infty \in \bar{B}_1(0)$. From Homework 7.2, it follows $\varphi_{z_0}(B_1(0)) = B_1(0)$, so that from continuity we deduce $|\varphi_{z_0}(z)| \leq 1$ for every $z \in \bar{B}_1(0)$. This contradicts the fact that $|\varphi_{z_0}(z_\infty)| \geq 1 + r$, which follows again from continuity. Thus $h \in \mathcal{H}(\bar{B}_1(0))$ as claimed.

The equality $h(B_1(0)) = f(B_1(0))$ follows from the fact shown in Homework 7.2 that $a\varphi_{z_0}$ maps $B_1(0)$ bijectively to $B_1(0)$.

Finally, a direct computation yields $\varphi'_{z_0}(0) = 1 - |z_0|^2$, so that by the chain rule

$$|h'(0)| = |f'(a\varphi_{z_0}(0))a\varphi'_{z_0}(0)| = |f'(-az_0)|(1 - |z_0|^2).$$

- b) Since $|f'(z)|(1 - |z|^2)$ depends continuously on $z \in \bar{B}_1(0)$, this function has a maximizer on $\bar{B}_1(0)$. Since this map vanishes on $\partial B_1(0)$ and is positive e.g. in 0, this maximum is achieved at an interior point $q \in B_1(0)$. Setting $F = f \circ \varphi_{-q}$, note that

$$\mathcal{F} = \{F \circ (a\varphi_{z_0}) : a \in \partial B_1(0), z_0 \in B_1(0)\}.$$

Indeed, for every $a_1, a_2 \in \partial B_1(0)$ and $z_1, z_2 \in B_1(0)$ the function $(a_1\varphi_{z_1}) \circ (a_2\varphi_{z_2})$ is again a biholomorphic map from $B_1(0)$ to $B_1(0)$ and in Homework 7.2 we have shown that such biholomorphic maps can always be represented in the form $a_3\varphi_{z_3}$ for some $a_3 \in \partial B_1(0)$ and $z_3 \in B_1(0)$.

⁴The optimal constant (say B) in Bloch's theorem is not known. There are lower and upper bounds. The best known thus far are

$$0.4332127... = \frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} \leq B \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)} = 0.4718617...$$

Thus part a. applied also to F implies that for every $z \in B_1(0)$ there exists $z' \in B_1(0)$ and $a \in \partial B_1(0)$ such that

$$\begin{aligned} |F'(z)|(1 - |z|^2) &= |(F \circ (-\varphi_z))'(0)| \\ &= |(f \circ (a\varphi_{z'}))'(0)| \\ &= |f'(-az')|(1 - |az'|^2) \\ &\leq |f'(q)|(1 - |q|^2), \end{aligned}$$

where in the last inequality we used the maximality of q . This concludes the proof of b.

c) By item a. we have that

$$F'(0) = |f'(q)|(1 - |q|^2),$$

so that b. implies $|F'(z)| \leq 2|F'(0)|$ whenever $|z| \leq \sqrt{2}/2$. Since $F'(0) \geq f'(0) = 1$ we know F is nonconstant. Hence Step 2 of the proof of Bloch's theorem and a. yield the inclusions $B_R(F(0)) \subset F(B_{\sqrt{2}/2}(0)) \subset F(B_1(0)) = f(B_1(0))$ for $R = 3\sqrt{2}/2 - 2$. This is the claim.

Homework 7.4 (Optional and difficult exercise). Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \rightarrow \mathbb{C}$ be holomorphic. We define $P := f(G) \cap \{-1, 1\}$. Show there exists a holomorphic function $g : G \rightarrow \mathbb{C}$ such that $f = \cos(g)$ if and only if for each $z_0 \in P$ the function $f - f(z_0)$ has a zero of even order⁵.

Solution. Assume first that for each fixed $z_0 \in P$ the map $f - f(z_0)$ has a zero of even order. Set $P_{\pm} = \{z \in G : f(z) = \pm 1\}$. We claim that we can define holomorphic square roots of $f(z) - 1$ and $f(z) + 1$ on G . Indeed, it follows from the Weierstraß product theorem that there exists a holomorphic function $h : G \rightarrow \mathbb{C}$ such that $\{h = 0\} = P_+$ and the multiplicity of each zero of h coincides with half the multiplicity of the zero of $f(z) - 1$ in P_+ (this is still an integer). Then the function $(f - 1)/h^2$ never vanishes on G , so that there exists $\tilde{h} : G \rightarrow \mathbb{C}$ holomorphic such that

$$f(z) - 1 = e^{\tilde{h}(z)} h(z)^2 = [e^{\tilde{h}(z)/2} h(z)]^2,$$

where we used G is simply connected.

By the same argument we can define a holomorphic square root of $f + 1$.

Next, we note the holomorphic assignment

$$f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1}$$

never vanishes on G , since otherwise $f(z)^2 = f(z)^2 - 1$, which is impossible. Hence we can define the holomorphic function

$$g(z) = -i \log(f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1}).$$

Then by the definition of the cosine we have

$$\begin{aligned} \cos(g(z)) &= \frac{1}{2} \exp(-i^2 \log(f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1})) \\ &\quad + \frac{1}{2} \exp(i^2 \log(f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1})) \\ &= \frac{1}{2} \left[f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1} + \frac{1}{f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1}} \right] \\ &= f(z). \end{aligned}$$

This proves the first claim.

Now suppose there exists $z_0 \in P_+$ such that $f - 1$ has a zero of odd multiplicity. Then we can write $f(z) - 1 = (z - z_0)^{2k-1} h(z)$ with $h : G \rightarrow \mathbb{C}$ holomorphic and $h(z_0) \neq 0$ and

⁵**Hint.** Show one can define the assignment $g(z) := -i \log(f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1})$. To define the square root, the Weierstraß product theorem might help.

$k \in \mathbf{N}$. Assume by contradiction that $f = \cos(g)$ for some holomorphic function $g : G \rightarrow \mathbf{C}$. Without loss of generality we may assume that $g(z_0) = 0$ with multiplicity $m \in \mathbf{N}$. Since $\cos(z) - 1 = z^2 k(z)$ with some holomorphic function $k : \mathbf{C} \rightarrow \mathbf{C}$ with $k(0) \neq 0$ it follows that $\cos(g(z)) - 1$ as a zero of order $2m$, which is even. This gives a contradiction. The case when $f + 1$ has a zero of odd multiplicity can be treated in a similar way using that without loss of generality $g(z_0) - \pi = 0$ with multiplicity $n \in \mathbf{N}$ and $\cos(z) + 1 = (z - \pi)^2 b(z)$ with $b : \mathbf{C} \rightarrow \mathbf{C}$ holomorphic and $b(\pi) \neq 0$, so that $\cos(g(z)) + 1$ has a zero with multiplicity $2n$.